Propagation of Uncorrelated Uncertainties

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1 The world's briefest calculus review

Partial derivatives are indicated by the "del" symbol ∂. For purposes of this class, you can think of it as a full differential (i.e.: $\frac{df}{dx}$ instead of $\frac{\partial f}{\partial x}$) if it helps.

Here are some simple examples to remind you how it's done:

$$C = A \quad \Rightarrow \quad \frac{\partial C}{\partial A} = 1$$

$$C = A + B \quad \Rightarrow \quad \frac{\partial C}{\partial A} = 1$$

$$C = AB \quad \Rightarrow \quad \frac{\partial C}{\partial A} = B$$

$$C = A^n B \quad \Rightarrow \quad \frac{\partial C}{\partial A} = nA^{n-1}B$$

$$C = \sin A \quad \Rightarrow \quad \frac{\partial C}{\partial A} = \cos B$$

If you need to take more complicated derivatives than this, you can look them up in any number of places (or do them numerically on a computer).

2 Propagation of uncertainties

If we measure independent quantities A and B that have respective uncertainties of σ_A and σ_B, we can then calculate a new quantity $C(A, B)$. The uncertainty on $C(A, B)$ is

$$\sigma_C^2 = \left(\frac{\partial C}{\partial A}\right)^2 \sigma_A^2 + \left(\frac{\partial C}{\partial B}\right)^2 \sigma_B^2 \tag{1}$$

This is simply extended for more than two independent measurements; just tack on more terms:

$$\sigma_C^2 = \left(\frac{\partial C}{\partial A}\right)^2 \sigma_A^2 + \left(\frac{\partial C}{\partial B}\right)^2 \sigma_B^2 + \left(\frac{\partial C}{\partial D}\right)^2 \sigma_D^2 + \cdots \tag{2}$$

One practical note: since we are assuming we are measuring independent quantities, the cross-terms ("covariances") are zero:

$$\frac{\partial B}{\partial A} = \frac{\partial A}{\partial B} = 0 \tag{3}$$

2.1 Example I: Velocity

A lab group (using a standard ruler with millimeter markings) measures the length of an air cart as $3.00 \pm .05$ cm. It passes through a photogate which records a transit time of $.205 \pm .001$ s. What is the velocity and its uncertainty?

Answer: The easy part: since $v = x/t$, $v = 3.00/.205 = 14.63$. I've extended our measurement to four significant digits since we're doing a detailed uncertainty analysis. Also note that I've tried to spell out the algebra step by step for this particular example; I don't need to see this level of detail in your derivation! Let's start off with Equation 1:

$$\sigma_v^2 = \left(\frac{\partial v}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial v}{\partial t}\right)^2 \sigma_t^2$$

and let's calculate the partial derivatives:

$$\frac{\partial v}{\partial x} = \frac{\partial(xt^{-1})}{x} = t^{-1}$$

$$\frac{\partial v}{\partial t} = \frac{\partial(xt^{-1})}{t} = -xt^{-2}$$

plugging these in directly above we get

$$\begin{aligned}
\sigma_v^2 &= (t^{-1})^2\sigma_x^2 + (-xt^{-2})^2\sigma_t^2 \\
&= \frac{\sigma_x^2}{t^2} + \frac{x^2\sigma_t^2}{t^4}
\end{aligned}$$

You could stop and plug in all the numbers here, but because $v = x/t$ we can put this expression in a more symmetric form:

$$\begin{aligned}
\sigma_v^2 &= \frac{\sigma_x^2}{x^2}\frac{x^2}{t^2} + \frac{\sigma_t^2}{t^2}\frac{x^2}{t^2} \\
&= \frac{\sigma_x^2}{x^2}v^2 + \frac{\sigma_t^2}{t^2}v^2 \\
\frac{\sigma_v^2}{v^2} &= \frac{\sigma_x^2}{x^2} + \frac{\sigma_t^2}{t^2} \qquad (4)
\end{aligned}$$

Equation 4 is a convenient form for memorizing or writing down. Now we can plug in values for x, v, t, σ_x, and σ_t:

$$\begin{aligned}
\frac{\sigma_v^2}{(14.6)^2} &= \left(\frac{.05}{3.00}\right)^2 + \left(\frac{.001}{.205}\right)^2 \\
\sigma_v^2 &= .060 \text{ cm}^2/\text{s}^2 \\
\sigma_v &= .25 \text{ cm/s}
\end{aligned}$$

The answer is: $v = 14.63 \pm .25$ cm/s. Also perfectly acceptable would be one with one less significant digit: $v = 14.6 \pm .3$ cm/s.

2.2 Example II: Circumference

(The detailed algebra in this one is left as an exercise for the reader.) Another lab group is measuring the circumference of a sheet of notebook paper. With their precision ruler, they measure the length $\ell = 20.96 \pm .05$ cm and the width $w = 29.21 \pm .05$ cm. What is the circumference and its uncertainty?

Answer: The easy part: recall that for a rectangle, $C = 2\ell + 2w$; plugging in we get $C = 100.34$ cm. Using Equation 1 we get

$$\sigma_C^2 = 4\sigma_\ell^2 + 4\sigma_w^2 \qquad (5)$$

And we can insert $\sigma_\ell = \sigma_w = .05$ cm to get $\sigma_C = .14$ cm. The overall answer: $C = 100.34 \pm .14$ cm.

References

[1] Bevington, P. R. and D. K. Robinson, *Data Reduction and Error Analysis for the Physial Sciences*, 1992.